ON STEINER POINTS OF CONVEX BODIES

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ABSTRACT

If s is a mapping from the set of all convex bodies in Euclidean space E^d to E^d which is additive (in the sense of Minkowski), equivariant with respect to proper motions, and continuous, then $s(K)$ is the Steiner point of the convex body K.

1. The Steiner point of a convex body, known for over a century, though not so extensively studied in former years, seems to have been reborn when Grünbaum [6, p. 238–239] noticed one of its important properties and posed a (still unsolved) problem on its characterization. Since then, quite a number of papers ([1], [5], [7 (14.3)], [8], [9], [10], [12], [13], [15], [16]) appeared which treated this remarkable point under different aspects. It is a variant of the original question of Grünbaum which makes the main object of the present paper. In an appendix we shall prove that centrally symmetric convex bodies can be characterized by a certain property of the Steiner points of their projections.

Let $\mathfrak{K}^d$ denote the set of all convex bodies (non-empty, compact, convex point sets) of d-dimensional Euclidean vector space E^d ($d \geqq 2$); let $\mathfrak{K}^d$, as usual, be equipped with Minkowski addition and Hausdorff metric. The Steiner point $s(K)$ of a convex body $K \in \mathfrak{K}^d$ may be defined by means of the following equation

$$s(K) = d \int_{S^{d-1}} uH(K,u)\, d\omega. \tag{1.1}$$

Here $S^{d-1} = \{u \in E^d \mid \| u \| = 1\}$ denotes the unit sphere of E^d, ω is the measure on S^{d-1} proportional to Lebesgue measure and satisfying $\omega(S^{d-1}) = 1$; $H(K, \cdot)$ is the support function of K. Evidently, the mapping $s\colon \mathfrak{K}^d \to E^d$ defined by 1.1 enjoys the following properties:

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(1.2) $\quad s(K_1 + K_2) = s(K_1) + s(K_2)$ for $K_1, K_2 \in \mathfrak{K}^d$,

(1.3) $\quad s(AK) = As(K)$ for $K \in \mathfrak{K}^d$ and each similarity $A: E^d \to E^d$,

(1.4) $\quad s$ is continuous.

Grünbaum's problem mentioned above consists in deciding whether s is characterized by 1.2 and 1.3 among all maps from $\mathfrak{K}^d$ to E^d. Whereas this seems to be difficult, the answer is in the affirmative if condition 1.4 is added. This has been proved, for $d = 2$, by Shephard [16]. Another proof, as well as an extension to $d = 3$, is due to Berg [1]. A proof by Schmitt [13] for arbitrary $d \geqq 2$ is erroneous, as was pointed out by Berg [1] and Meyer [10]. Hadwiger [8] states a more general* result which, however, cannot be true in this generality, as counter examples show (there is a mistake on p. 172, lines 8–6 from below; compare Hadwiger [9]). Recently Meyer [10] has shown that the Steiner point is characterized by conditions 1.2, a weakened form of 1.3, and a strengthened form of 1.4, namely with uniform continuity instead of mere continuity. In the following, we shall give a proof for arbitrary dimension $d \geqq 3$, using 1.2, a weakened form of 1.3, and 1.4.

2. We shall prove

THEOREM 1. *Let $f: \mathfrak{K}^d \to E^d$ be a mapping with the following properties:*

(2.1) $\quad f(K_1 + K_2) = f(K_1) + f(K_2)$ *for all* $K_1, K_2 \in \mathfrak{K}^d$,

(2.2) $\quad f(TK) = Tf(K)$ *for each* $K \in \mathfrak{K}^d$ *and each proper motion* $T: E^d \to E^d$,

(2.3) $\quad f$ *is continuous.*

Then $f = s$.

It is easy to see that condition 2.1 together with 2.3 implies

(2.4) $\quad f(\mu K) = \mu f(K)$ for each $K \in \mathfrak{K}^d$ and each real $\mu \geqq 0$.

We wish to point out that the equivariance property $f(TK) = Tf(K)$ is supposed only for proper motions T. In spaces of odd dimension it is, therefore, not a priori clear that for a centrally symmetric convex body $K \in \mathfrak{K}^d$ the point $f(K)$ necessarily coincides with the center (and hence with the Steiner point) of K. Because of this fact we cannot utilize asymmetry classes of convex bodies, as was done in [13]

* Relation 2.1 implies additivity in Hadwiger's sense, since $K_1 \cup K_2 + K_1 \cap K_2 = K_1 + K_2$, if K_1, K_2, and $K_1 \cup K_2$ are convex (see Sallee [12], p. 77, where the corresponding result for support functions is proved).

and [16]. Nevertheless our proof is, in a certain sense, similar to Shephard's [16].

In the following we may restrict ourselves to the case $d \geqq 3$ since it is easy to see that Shephard's proof for $d = 2$ works also under the weaker assumption 2.2. In order to prove Theorem 1 it will be sufficient to prove the equality $f(K) = s(K)$ for each K contained in a dense subset of $\mathfrak{K}^d$; the general result then follows from the continuity of s and f (this will be the only step where the continuity of f is needed). A suitable dense subset of $\mathfrak{K}^d$ is obtained as follows.

By *a spherical harmonic* (of dimension d) of degree m we understand, as usual, the restriction to S^{d-1} of a real function, defined on E^d, which in Cartesian coordinates is expressed as a harmonic polynomial, homogeneous of degree m ($m = 0, 1, 2, \cdots$). Let $\mathscr{H}$ denote the real vector space whose elements are the finite sums of spherical harmonics. Let $\mathfrak{K}(\mathscr{H}) \subset \mathfrak{K}^d$ be the set of those convex bodies whose support function, restricted to S^{d-1}, belongs to $\mathscr{H}$.

LEMMA 1. *$\mathfrak{K}(\mathscr{H})$ is dense in $\mathfrak{K}^d$.*

PROOF. Let G be a twice continuously differentiable function on S^{d-1}. We extend G to a function on $E^d - \{0\}$ by defining $G(x) = \| x \| G(x/\| x \|)$ for $x \in E^d - \{0\}$. Thus G is positively homogeneous of degree one. For $x \in E - \{0\}$ write

$$G_{ik}(x) = \frac{\partial^2 G(x)}{\partial x_i \partial x_k},$$

where $x_1, \cdots, x_d$ are Cartesian coordinates. By homogeneity, the matrix $(G_{ik}(x))$ has 0 as an eigenvalue (with corresponding eigenvector x). Of the remaining eigenvalues, let $\eta(G, x)$ denote the smallest. Put

$$\eta(G) = \min_{\|x\| = 1} \eta(G, x);$$

the minimum is attained since $\eta(G, x)$ is continuous on $E^d - \{0\}$. Now it is well known that the function G is convex and hence a support function if, and only if, the quadratic form $\Sigma G_{ik}(x) \alpha^i \alpha^k$ is everywhere positive semidefinite. The latter condition is equivalent to $\eta(G) \geqq 0$.

Let $K \in \mathfrak{K}^d$ be a convex body whose support function is analytic and whose boundary has everywhere positive radii of curvature. Since the set of such bodies is dense in $\mathfrak{K}^d$, Lemma 1 will be proved if we show that K can be approximated by bodies contained in $\mathfrak{K}(\mathscr{H})$. Now the support function H (restricted to S^{d-1}) of K can be expressed as a uniformly convergent series of spherical harmonics.

Hence there exists a sequence $H_1, H_2, \cdots$ of functions contained in $\mathscr{H}$ which converge, uniformly on S^{d-1}, to the function H. Let us extend the functions H, H_1, $H_2, \cdots$ to $E^d - \{0\}$ by positive homogeneity of degree one. Then the relation $\lim_{j\to\infty} H_j = H$ holds uniformly in every compact subset of $E^d - \{0\}$. From the analyticity of H we may deduce that this uniform convergence carries over to the partial derivatives (with respect to Cartesian coordinates) of any order of the functions in question. We conclude that

$$\lim_{j\to\infty} \eta(H_j) = \eta(H).$$

By assumption we have $\eta(H) > 0$, since the eigenvalues of the matrix $(H_{ik}(x))$ for $\| x \| = 1$ are, besides zero, just the main radii of curvature of the boundary of K in the point with outer normal vector x (see Bonnesen-Fenchel [3], p. 61). Hence, for j sufficiently large, we have $\eta(H_j) > 0$, so that almost every function of the sequence $H_1, H_2, \cdots$ is a support function. This proves Lemma 1.

Let $f: \mathfrak{K}^d \to E^d$ be a mapping satisfying 2.1 and 2.4. It induces canonically a mapping $\bar{f}$: $\mathscr{H} \to E^d$, namely in the following way. Let $S \in \mathscr{H}$. The function S is twice continuously differentiable (even analytic). Let $H_R(x) = R\| x \|$ for $R \geqq 0$; thus H_R is the support function of the spherical ball B_R with radius R and center in the origin 0 of E^d. Since (with η as defined in the proof of Lemma 1)

$$\eta(S + H_R) = \eta(S) + R,$$

the function $S + H_R$ is the support function of a convex body $K_{S,R} \in \mathfrak{K}^d$ provided R is sufficiently large. Define

$$\bar{f}(S) = f(K_{S,R}) - f(B_R),$$

which makes sense since the righthand side does not depend on R, as follows at once from 2.1. The mapping $\bar{f}$: $\mathscr{H} \to E^d$ thus defined has the property

$$\bar{f}(S_1 + S_2) = \bar{f}(S_1) + \bar{f}(S_2) \text{ for } S_1, S_2 \in \mathscr{H},$$

which is an immediate consequence of 2.1. This leads to $\bar{f}(0) = 0$ and thus to $\bar{f}(-S) = -\bar{f}(S)$; this, in turn, shows that the relation $\bar{f}(\mu S) = \mu\bar{f}(S)$, valid for $\mu \geqq 0$ according to 2.4, holds for arbitrary real μ. Thus $\bar{f}$ is a linear map.

Let T: $E^d \to E^d$ be a rotation (with fixed point 0). For $S \in \mathscr{H}$ let TS denote the left translate of S by T, i.e. $(TS)(u) = S(T^{-1}u)$ for $u \in S^{d-1}$. We have $TS \in \mathscr{H}$.

LEMMA 2. *Let $d \geqq 3$. If $\bar{f}$: $\mathscr{H} \to E^d$ is a vector space homomorphism which satisfies $\bar{f}(TS) = T\bar{f}(S)$ for each proper rotation T of E^d and each $S \in \mathscr{H}$, then*

(2.5) $$\bar{f}(S) = \alpha \int_{S^{d-1}} uS(u)d\omega \text{ for each } S \in \mathscr{H}$$

with some constant real number α.

Let us first show that Theorem 1 is a consequence of Lemmas 2 and 1: If $f: \mathfrak{K}^d \to E^d$ is a mapping with properties 2.1, 2.2, 2.3, then the induced map $\bar{f}: \mathscr{H} \to E^d$ is a vector space homomorphism which, as a consequence of 2.2 and 2.1, satisfies the assumptions of Lemma 2. Hence 2.5 holds with some α. By 2.2 the relation $f(TK) = Tf(K)$ is also true if $T: E^d \to E^d$ is a translation. Therefore, if $c \in E^d$ and $K \in \mathfrak{K}^d$, we have

$$f(K) + f(\{c\}) = f(K + \{c\}) = f(K) + c,$$

since the body $K + \{c\}$ is obtained from K by translation by the vector c. We conclude that $f(\{c\}) = c$. The support function of the convex body which contains only the point c is given by $S_c(u) = \langle c, u \rangle$; it belongs to $\mathscr{H}$. We deduce that

$$c = \bar{f}(S_c) = \alpha \int_{S^{d-1}} u\langle c, u\rangle d\omega.$$

But for any $c \in E^d$

(2.6) $$c = d \int_{S^{d-1}} u\langle c, u\rangle d\omega$$

holds, so that $\alpha = d$. Hence if $K \in \mathfrak{K}(\mathscr{H})$, and $H \in \mathscr{H}$ is the support function of K, then $f(K) = \bar{f}(H) = s(K)$ by 1.1. Then Lemma 1 together with 2.3 and 1.4 shows that $f(K) = s(K)$ for each $K \in \mathfrak{K}^d$.

Proof of Lemma 2. For $m = 0, 1, 2, \cdots$ let $\mathscr{H}_m$ denote the subspace of $\mathscr{H}$ which consists of all spherical harmonics of degree m. The dimension of $\mathscr{H}_m$ is given by

(2.7) $$\dim \mathscr{H}_m = \frac{2m + d - 2}{m + d - 2}\binom{m + d - 2}{m}$$

(see Müller [11], p. 4–5.) Under the assumptions of Lemma 2 we want to show that

(2.8) $$\bar{f}(S) = \begin{cases} \alpha \int_{S^{d-1}} uS(u)d\omega & \text{for } S \in \mathscr{H}_1, \\ 0 & \text{for } S \in \mathscr{H}_m, \quad m \neq 1, \end{cases}$$

where α is a constant. This will prove Lemma 2, since each element of $\mathscr{H}$ is a finite sum of elements of $\bigcup_{m=0}^{\infty} \mathscr{H}_m$, and since the relation

$$\int_{S^{d-1}} uS(u)d\omega = 0 \qquad \text{for } S \in \mathscr{H}_m, \quad m \neq 1,$$

holds. This equation is one of the orthogonality relations for spherical harmonics (observe that, for fixed $c \in E^d$, the function $\langle c, u\rangle$ is an element of $\mathscr{H}_1$).

We proceed to the proof of 2.8. First let $S \in \mathscr{H}_0$. Then S is a constant function, so that $TS = S$ for each rotation T. The assumption $\bar{f}(TS) = T\bar{f}(S)$ shows that $\bar{f}(S)$ is invariant under all rotations and hence is the zero vector. This proves 2.8 for $m = 0$.

Now define a map $A: E^d \to E^d$ by the equation

$$Ax = \bar{f}(S_x),$$

where $S_x(u) = \langle x, u\rangle$. Evidently, A is linear. Let T be a proper rotation of E^d. We have

$$S_{Tx}(u) = \langle Tx, u\rangle = \langle x, T^{-1}u\rangle = S_x(T^{-1}u) = (TS_x)(u)$$

and therefore, by the definition of A and the assumption of Lemma 2,

$$ATx = \bar{f}(S_{Tx}) = \bar{f}(TS_x) = T\bar{f}(S_x) = TAx.$$

Since $x \in E^d$ is arbitrary, we see that the linear map A commutes with each proper rotation. Because of the assumption $d \geqq 3$, this is only possible if A is a dilatation by some factor α/d (perhaps zero). Now let $S \in \mathscr{H}_1$; then $S(u) = \langle c, u\rangle$ for some constant vector $c \in E^d$. Therefore we have

$$\bar{f}(S) = Ac = (\alpha/d)c = \alpha \int_{S^{d-1}} u\langle c, u\rangle d\omega = \alpha \int_{S^{d-1}} uS(u)d\omega,$$

where 2.6 was used. This proves the first part of 2.8.

Let T: $E^d \to E^d$ be a proper rotation. Since $S \in \mathscr{H}_m$ implies $TS \in \mathscr{H}_m$, a map C_T: $\mathscr{H}_m \to \mathscr{H}_m$ is defined if we put $C_T(S) = TS$. It is obvious that this map is linear, and that it is orthogonal if a scalar product $\langle\ ,\ \rangle$ on $\mathscr{H}_m$ is defined by

$$\langle S_1, S_2\rangle = \int_{S^{d-1}} S_1 S_2 d\omega, \qquad S_1, S_2 \in \mathscr{H}_m.$$

Moreover, we have $C_{RT} = C_R C_T$, so that the mapping $T \to C_T$ is an orthogonal representation of the rotation group $SO(d)$. We will make use of the well known fact that this representation is irreducible (see, e.g., Coifman and Weiss [4], p. 136).

Consider the kernel ker $\bar{f}_m$ of the vector space homomorphism $\bar{f}_m$: $\mathscr{H}_m \to E^d$, where $\bar{f}_m$ denotes the restriction of $\bar{f}$ to $\mathscr{H}_m$. If $S \in \ker \bar{f}_m$ and T is a proper rotation of E^d, then by the definition of C_T and the assumption of Lemma 2,

$$\bar{f}_m(C_T(S)) = \bar{f}_m(TS) = T\bar{f}_m(S) = 0,$$

which shows that $C_T(S) \in \ker \bar{f}_m$. Thus $\ker \bar{f}_m$ is an invariant subspace of the representation $T \to C_T$. Since this representation is irreducible, $\ker \bar{f}_m$ is either $\{0\}$ or the whole space $\mathscr{H}_m$. If we now assume $m \geqq 2$ (and $d \geqq 3$), then $\dim \mathscr{H}_m > d$ by 2.7, so that $\ker \bar{f}_m$ has positive dimension and hence coincides with $\mathscr{H}_m$. This proves $\bar{f}(S) = 0$ for each $S \in \mathscr{H}_m$, which completes the proof of 2.8 and hence that of Lemma 2.

3. Appendix. Here we wish to prove a simple theorem which contains a characterization of centrally symmetric convex bodies. This theorem may be viewed as a counterpart to the following one which is due (at least for $d = 3$) to Blaschke [2] and which may be proved in an analogous way: *Let $K \in \mathfrak{K}^d$ be a convex body with interior points, and let $p \in K$ be a point such that the centroids (with respect to a homogeneous mass distribution) of all sections of K by hyperplanes through p coincide with p. Then K has p as its center.*

If $E \subset E^d$ is a hyperplane, let us denote by π_E the orthogonal projection onto E.

THEOREM 2. *Let $K \in \mathfrak{K}^d$ be a convex body and suppose that there exists a point $p \in K$ such that, for any hyperplane $E \subset E^d$, the Steiner point of $\pi_E K$ coincides with $\pi_E p$. Then K is centrally symmetric with respect to p.*

PROOF. For $d = 2$ the assertion is trivial; so let us assume $d \geqq 3$. Because of property 1.3 of the Steiner point we may suppose that p is equal to the origin 0 of E^d. For $u \in S^{d-1}$ let K_u denote the convex body that arises by orthogonal projection of K on to the hyperplane through 0 with normal vector u. Then the assumption of Theorem 2 is equivalent to

$$s(K_u) = 0 \quad \text{for each } u \in S^{d-1}.$$

he representation 1.1, applied in the linear subspace orthogonal to u, yields

$$\int_{\sigma_u} vH(K, v)d\lambda = 0 \quad \text{for each } u \in S^{d-1}. \tag{3.1}$$

Here

$$\sigma_u = \{v \in S^{d-1} \mid \langle v, u \rangle = 0\},$$

and λ denotes the $(d-2)$-dimensional Lebesgue measure on the great sphere σ_u. We have used the obvious relation

$$H(K_u, v) = H(K, v) \quad \text{for } \langle v, u\rangle = 0$$

for the support function H. Equation 3.1 is equivalent to the following one:

$$(3.2) \quad \int_{\sigma_u} v[H(K,v) + H(K,-v)]d\lambda + \int_{\sigma_u} v[H(K,v) - H(K,-v)]d\lambda = 0.$$

Here the first integral is equal to zero since the integrand is odd. The integrand of the second integral is even, therefore we can apply the following lemma (after multiplying both sides of 3.2 with an arbitrary constant vector):

LEMMA 3. *If g is an even, continuous, real function on S^{d-1} which satisfies*

$$\int_{\sigma_u} g d\lambda = 0 \textit{ for each } u \in S^{d-1},$$

then $g \equiv 0$.

We deduce that

$$v[H(K,v) - H(K,-v)] = 0 \text{ for each } v \in S^{d-1},$$

hence $H(K,v) = H(K,-v)$, from which the assertion of Theorem 2 follows.

Lemma 3 is, for $d = 3$, essentially due to Minkowski (see Bonnesen-Fenchel [3], p. 136–138). For further references, as well as for a general proof, see [14].

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